Unsteady Radiative Heat Transfer in a Scattering-Dominant Medium

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A theoretical investigation is made of the unsteady radiative heat transfer through a scattering-dominant absorbing emitting gray medium that occupies a semi-infinite space bounded by a semitransparent gray plate. The system is initially in a uniform state. Simultaneously, the temperature of the plate is changed to a constant value and a beam of radiation is imposed through the plate. The temperature and radiation levels are maintained thereafter. It is found that the radiation field spreads deeply due to strong scattering. A temperature rise follows the spreading by a weak net absorption. Consequently, the behavior of the temperature is qualitatively different from that in a nonscattering medium.

Nomenclature

A_i, B_{en}, B_{mn} = numerical constants
$\hat{c}$ = specific heat of the medium
E_n = exponential integral of the nth order
F_i = function representing the form of the external radiation, defined in Eq. (18)
f_i = representative quantity of i_i, T_i, or q_i
$\mathcal{G}_e, \mathcal{G}_n$ = known functions of corrections in the radiation layer
$\mathcal{H}, \mathcal{I}$ = functions in Eqs. (19) and (A3)
I_0, I_1 = modified Bessel function of the first kind of order zero and one
i = radiative intensity
$\mathfrak{M}^H, \mathfrak{N}^H$ = functions of a part of $\mathcal{I}^H$
n = refractive index
p = parameter of the Laplace transform
q = heat flux
S_n = numerical constants
T = temperature
t = $\epsilon^2\tau/3$, reduced time
x = $\epsilon\xi$, reduced spatial coordinate
$\hat{\alpha}$ = extinction coefficient
β = $\omega\epsilon^2/[3(1-\omega)]$
γ = $\sqrt{p/(\beta p+1)}$
δ = delta function or Kronecker's delta
ϵ = dummy parameter representing the ratio of the photon mean free path against a standard length of the asymptotic region
Θ_n = function defined in Eq. (20)
θ_w = transmittance of the boundary plate
Λ = integral operator
μ, ν = direction cosines of a ray
ξ = distance from the boundary plate
π = ratio of the circumference of a circle to its diameter
$\hat{\rho}$ = density of the medium
$\hat{\sigma}$ = Stefan-Boltzmann constant
τ = time
φ = $(1-\omega)\tau/(2\omega)$
ω = albedo of scattering

Subscripts

b = beam
i = effects due to the beam ($i=1$) or the isotropic radiation ($i=2$)
j = order of expansions with respect to ϵ
m,n = ordering subscripts
s = equilibrium state
w = boundary plate

Superscripts

H = asymptotic region
K = correction in the radiation layer
$(\hat{\ })$ = dimensional quantities
$(\bar{\ })$ = image of Laplace transform
$(\ )'$ = integration variables

Introduction

DEVELOPMENTS in many fields of engineering force consideration of higher and higher temperatures, which in turn requires a more detailed knowledge of radiative heat transfer. One such area is radiation gasdynamics. The aim of this paper is to increase the fundamental understanding of radiative heat transfer.

When a radiating medium is nonscattering, the medium must be heated to a very high temperature for the radiative heat transfer to be dominant in comparison to the heat conductivity. Most practical problems (except the planetary re-entry) will occur in a lower temperature range than this. If the medium is scattering dominant, the effect of thermal radiation will be considerable in this low-temperature range. For example, the radiation effect will be important in the nozzle flow of a rocket using a solid propellant having a number of small particles of a metal that scatters radiation. In a high-temperature furnace, there are many particles of burned fuel, dust, and impurities. Also, there are many scattering particles in an ablated layer of a re-entry body. Therefore, there are many possible applications of a scattering-dominant radiative heat transfer.

When the radiative heat transfer is steady, the transfer with isotropic scattering can be regarded as a nonscattering one, if the extinction coefficient is considered to be the effective absorption coefficient.[1,2]

Isotropic scattering is a simple model of natural phenomena, although it has wide applicability in many problems. There are a number of reports on the effects of real anisotropic scattering (e.g., Bergstrom and Cogley[3]). A detailed study of such scattering is not considered in this paper.

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In recent years, unsteady problems have been investigated. For example, Kubo and Hayashiguchi[4] made a study of unsteady radiative heat transfer in a nonscattering medium, and Ganapol and Pomraning[5] studied a problem that included the effect of the unsteady term in the radiative transfer equation on a nonscattering medium.

However, in an unsteady problem, scattering may cause a qualitative change in the characteristics of the radiative heat transfer, although it can be regarded as a process of similarity in a steady problem. The reason is that the net radiative absorption is smaller than that of the nonscattering case. Therefore, the temperature rise will be slower than that in the nonscattering medium. The effect of the isotropic scattering was investigated by Kubo and Hayashiguchi[6] in an unsteady radiative heat-transfer situation where the albedo of scattering was less than about 0.8. The results showed that the effect of scattering could be practically regarded as a similarity effect in this range of the albedo of scattering.

By contrast, when the medium is scattering-dominant (e.g., $0.8 \sim 1$ of the albedo of scattering), the analysis of Ref. 6 is no longer valid. The reason is that in Ref. 6 the effect of scattering was investigated as a direct extension of nonscattering radiative heat transfer. The radiation field was restricted in the radiation layer, the thickness of which was of the order of the photon mean free path adjacent to the boundary in an initial period. However, in the final period, the radiation from the boundary penetrates deeply due to the irradiation from the medium itself. In a scattering-dominant medium, the net absorption and re-emission is naturally very weak. The radiation field instantaneously reaches its local equilibrium state, since the speed of light is assumed to be infinitely large. This means that the effect of the radiation imposed at the boundary spreads instantaneously over the entire region. A slow rise in the temperature is caused by a weak net absorption. Therefore, the time scale of the temperature change is very small in spite of an abrupt change in the imposed radiation. Thus, not only the radiation layer, but also an asymptotic region from the initial stage of the process, should be considered. The scheme of the transfer process is quite different from that of a nonscattering medium. Therefore, the problem of unsteady radiative heat transfer in a scattering-dominant medium is also interesting from the theoretical point of view. In this paper, we will try to consider a simple problem with a rigorous treatment rather than solve a practical problem having complicated effects.

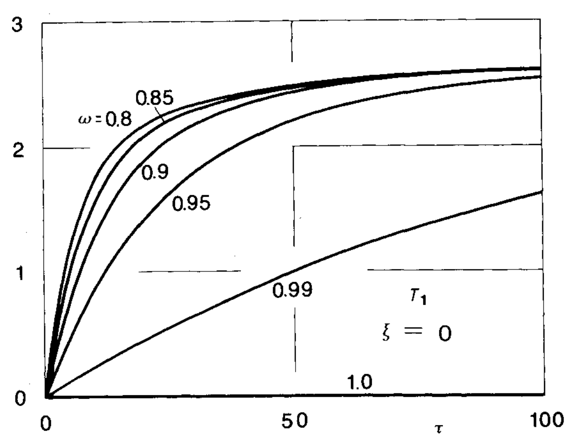

Fig. 1a Change of temperature at the boundary for the beam radiation $T_1(0,\xi)$ for various values of ω.

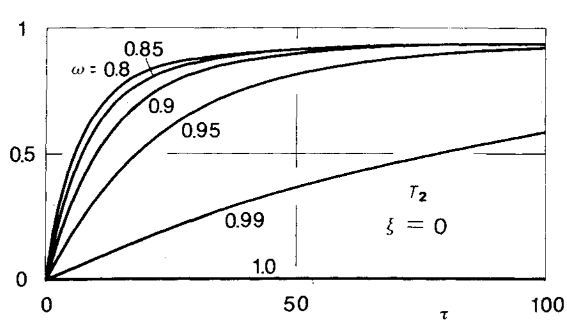

Fig. 1b Change of temperature at the boundary for the blackbody radiation $T_2(0,\xi)$ for various values of ω.

To study this effect as simply as possible, we consider the following problem. A scattering-dominant absorbing emitting medium occupies a semi-infinite space bounded by a semitransparent plate. At first, the system is in a uniform state. Instantaneously, the temperature of the boundary plate is changed to a constant value that is maintained thereafter. At the same time, an external radiation is imposed as a beam normal to the boundary. The changes are small enough to linearize the equations concerned. The motion, thermal expansion, and conductivity of the medium are neglected. The scattering is isotropic with an albedo close to one. The medium and the plate are gray. They are in a local thermodynamic equilibrium.

Many of these assumptions are not essential. However, without them, our analysis would be more complicated, although essentially unchanged. We will discuss this in the last section.

Basic Equations, Initial and Boundary Conditions

For the problem mentioned above, the conservation equation of energy is

$$\hat\rho \hat c \frac{\partial \hat T}{\partial \hat\tau} + \frac{\partial \hat q}{\partial \hat\xi} = 0 \tag{1}$$

where $\hat\rho$, $\hat T$, and $\hat c$ are the density, temperature, and specific heat of the medium, respectively. The notation (^) denotes quantities with dimensions. The distance $\hat\xi$ is measured from the boundary and $\hat\tau$ is the time measured from the instance of the changes at the boundary plate. The radiative heat flux $\hat q$ depends on the radiative intensity $\hat i(\hat\xi, \hat\tau, \mu)$ by the relation

$$\hat q = 2\pi \int_{-1}^{1} \mu' \hat i \, d\mu' \tag{2}$$

where μ is the direction cosine of a ray in the $\hat\xi$ direction. The radiative transfer equation for a gray medium with isotropic scattering is

$$\mu \frac{\partial \hat i}{\partial \hat\xi} = -\hat\alpha \left[\hat i - \frac{1-\omega}{\pi} \hat\sigma n^2 \hat T^4 - \frac{\omega}{2} \int_{-1}^{1} \hat i(\mu') \, d\mu' \right] \tag{3}$$

where $\hat\alpha$, ω, and n are the extinction coefficient, albedo of scattering, and refractive index of the medium, respectively, and $\hat\sigma$ the Stefan-Boltzmann constant. For an unsteady problem, Eq. (3) should have the unsteady term. The time scale of the unsteady term is proportional to the inverse of the light speed. Therefore, we can neglect this term in almost all problems. The unsteadiness of the problem is caused by the unsteady term in Eq. (1).

When the problem is steady, $\partial \hat T / \partial \hat\tau = 0$, and thus $\int_{-1}^{1} \hat i \, d\mu' = 2\hat\sigma n^2 \hat T^4 / \pi$. Therefore, Eq. (3) can be rewritten in a form formally independent of ω. But this is impossible in an unsteady problem. This fact suggests that the scattering may qualitatively affect the radiative heat transfer.

The initial and boundary conditions are as follows. When $\hat\tau < 0$, our system is in a equilibrium state,

$$\hat i = \hat i_s, \qquad \hat T = \hat T_s, \qquad \hat q = 0 \tag{4}$$

where $\hat{i}_s$ and $\hat{T}_s$ are the radiative intensity and temperature, respectively, in the uniform state. Subscript s denotes quantities of the initial uniform state. When $\hat\tau \geqq 0$, the temperature of the boundary plate is maintained to a constant value $\hat{T}_w$ and, through the plate, a beam (whose equivalent temperature is $\hat{T}_b$) is imposed normally. The subscript w denotes quantities of the boundary plate and b those of the beam. Therefore, the radiative intensity of a ray directed into the medium at the boundary plate is

$$\hat{i}(0,\hat\tau\geqq 0,\mu>0) = \theta_w(\hat\sigma/\pi)\hat{T}_b^4\delta(\mu-1)\delta(\nu) + (1-\theta_w)(\hat\sigma/\pi)n_w^2\hat{T}_w^4 \tag{5}$$

where θ_w and n_w are transmittance and the refractive index of the plate, respectively, ν the direction cosine of the ray in a direction normal to the $\hat\xi$ axis, and δ the delta function. This boundary condition will be given somewhat arbitrarily. If we consider a solid medium, we need no boundary plate—we can apply an arbitrarily designed external radiation instead of the present boundary condition. The present condition was chosen only as a simple model of a typical case. It is easy to change this condition to some other condition.

In a far region, the medium is in the uniform state provided initially, even when $\hat\tau \geqq 0$.

$$\lim_{\xi\to\infty}(\hat{i},\hat{T},\hat{q}) = (\hat{i}_s,\hat{T}_s,0) \tag{6}$$

If $|\hat{T}_w - \hat{T}_s|/\hat{T}_s$ and $\hat{T}_b$ is small enough, we can linearize these equations. We introduce

$$\hat{T} = \hat{T}_s(1+T), \qquad \hat{q} = 16n^2\hat\sigma\hat{T}_s^4\cdot q \tag{7a}$$

$$\hat{i} = n^2(\hat\sigma/\pi)\hat{T}_s^4(1+16\pi i) \tag{7b}$$

$$\hat{T}_w = \hat{T}_s(n/n_w)^{1/2}(1-\theta_w)^{-1/4}(1+T_w) \tag{7c}$$

$$\hat{T}_b = \hat{T}_s(4n^2T_b/\theta_w)^{1/4} \tag{7d}$$

$$\hat\xi = \xi/\alpha, \qquad \hat\tau = \hat\rho\hat{c}\tau/(16n^2\hat\alpha\hat\sigma\hat{T}_s^3) \tag{7e}$$

Assmuming that the nondimensional quantities T, q, T_w, T_b, and i are small and substituting the above expressions into Eqs. (1-6) we obtain the linearized relations

$$\frac{\partial T}{\partial\tau} + \frac{\partial q}{\partial\xi} = 0 \tag{1'}$$

$$q = 2\pi\int_{-1}^{1}\mu' i\,\mathrm{d}\mu' \tag{2'}$$

$$\mu\frac{\partial i}{\partial\xi} = -i + \frac{1-\omega}{4\pi}T + \frac{\omega}{2}\int_{-1}^{1} i(\mu')\,\mathrm{d}\mu' \tag{3'}$$

$$i = T = q = 0, \qquad \text{when } \tau<0 \tag{4'}$$

$$i(0,\tau\geqq 0,\mu>0) = (1/4\pi)[T_b\delta(\mu-1)\delta(\nu) + T_w] \tag{5'}$$

$$\lim_{\xi\to\infty}(i,T,q) = 0, \qquad \text{when } \tau\geqq 0 \tag{6'}$$

Equation (5′) suggests that i, T, and q can be expressed in the form

$$f = (1/4\pi)T_b\cdot f_1 + T_w\cdot f_2 \tag{8}$$

where f represents one of i, T, or q. The separation of f, which is due to linearization, permits separate consideration of the effects of different types of boundary conditions. Thus, f_1 shows the effect of the normal beam radiation as f_2 the blackbody radiation. These two effects are qualitatively different even in a steady problem.[7] These equations and the initial and boundary conditions are the same as those considered in a previous paper.[6] We would remember that these equations already express a kind of similarity because the time and distance are normalized by using the extinction coefficient instead of the net absorption coefficient. In other words, a solution expressed by variables ξ and τ already has a kind of scattering effect. The results of the previous paper[6] show that the temperature distributions are effectively the same in these terms when $0\leqq\omega\lesssim 0.8$. Our problem is to solve the above equation when $0.8\lesssim\omega<1$.

When τ tends to infinity, our solution reaches its steady form. The steady solution was obtained in a previous paper.[7]

Analysis for a Scattering-Dominant Medium

If the medium is scattering-dominant, i.e., $\omega \doteqdot 1$, we find from Eq. (3′) that the net irradiation from the high-temperature medium $(1-\omega)T/(4\pi)$ has only a small effect. Therefore, the radiation field is primarily governed by scattering. The radiative transfer equation (3′) has no unsteady term. The radiation field has its local equilibrium form for each instance. This local equilibrium radiation field is almost homogeneous in space. A slightly inhomogeneous radiation field yields a weak heat flux that causes a slow temperature rise in the medium. However, the temperature rise spreads all over the spatial region. Consequently, we should consider not only the radiation layer but also an asymptotic region from the initial stage of the process.

According to the previous analyses,[4,6] the time scale of the temperature rise is of order ϵ^2, where ϵ is the ratio of the photon mean free path to a standard length scale in the asymptotic region. We consider the case where the small factor $(1-\omega)/\omega$ is also of order of ϵ^2 and make use of the method of matched asymptotic expansions (cf. Refs. 6-8).

At first t is introduced by

$$t = \epsilon^2\tau/3 \tag{9}$$

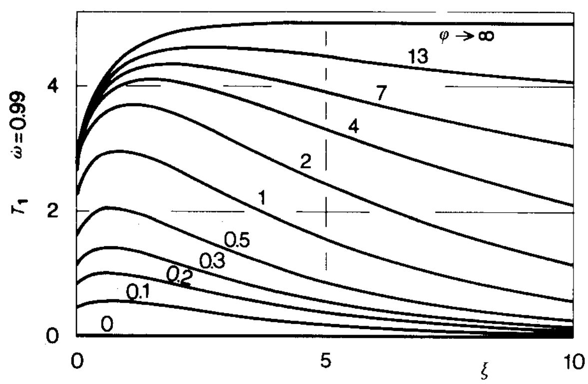

Fig. 2a Temperature profile for the beam radiation T_1 for various values of φ when $\omega = 0.99$.

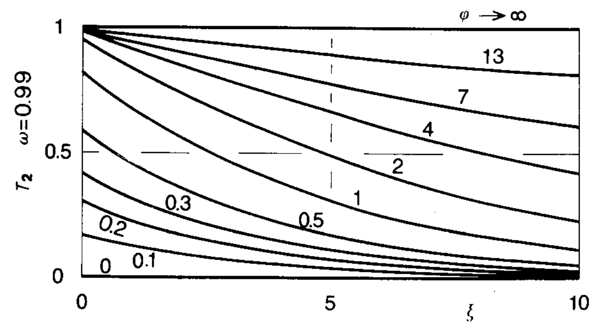

Fig. 2b Temperature profile for the blackbody radiation T_2 for various values of φ when $\omega = 0.99$.

All quantities are transformed by the Laplace transformation

$$\bar{f}_i(\xi,p)=\int_0^\infty f_i(\xi,t')e^{-pt'}\mathrm{d}t' \tag{10}$$

where $i=1$ or 2.

Asymptotic Field

In the asymptotic field the length scale is of order of $1/\epsilon$. We introduce a new spatial variable x by

$$x=\epsilon\xi \tag{11}$$

All quantities are expanded into a series

$$\bar{f}_i^H=\bar{f}_{i0}^H+\epsilon\bar{f}_{i1}^H+\epsilon^2\bar{f}_{i2}^H+\epsilon^3\bar{f}_{i3}^H+\epsilon^4\bar{f}_{i4}^H+\ldots \tag{12}$$

where the superscript H denotes quantities in the asymptotic region. Substituting the series into Eqs. (1′-4′) and (6′) and equating terms of the same order of ϵ, we finally obtain the relations

$$\frac{\mathrm{d}^2\bar{T}_{ij}^H}{\mathrm{d}x^2}-\gamma^2\bar{T}_{ij}^H=-\frac{3}{5}\gamma^4\bar{T}_{i,j-2}^H \quad \text{for } i=1 \text{ or } 2,\quad j=0\sim 3 \tag{13}$$

where $\gamma=\sqrt{p/(\beta p+1)}$ and $\beta=\omega\epsilon^2/3(1-\omega)$. $\bar{T}_{ij}^H$ is zero if $j<0$. The left-hand side of Eq. (13) is the primary part of the equation. The right-hand side gives only a correction to a higher-order equation by a lower-order quantity. The formal inverse Laplace transform of the primary part of the equation leads to, e.g.,

$$\frac{\partial T_{i0}^H}{\partial t}=\frac{\partial^2 T_{i0}^H}{\partial x^2}+\frac{\omega\epsilon^2}{3(1-\omega)}\frac{\partial^3 T_{i0}^H}{\partial t\partial x^2} \tag{14}$$

for $j=0$. This is the primary equation governing T_i^H in the asymptotic region. This equation differs from a simple diffusion relation in the last term. This fact shows us that the radiative heat transfer can no longer be approximated by a diffusion equation in an unsteady problem with a scattering-dominant medium. The effect of the new term will be covered by the small factor ϵ^2 when the scattering is weak or moderate. But it will be more and more significant when the scattering becomes stronger, a characteristic that does not depend on the linearization. Therefore, this gives a qualitative difference of a radiative heat transfer in a scattering-dominant medium.

The radiative intensity i corresponding to the temperature field governed by Eq. (13) is almost isotropic, as is easily shown by Eq. (3′). It cannot satisfy the boundary condition in Eq. (5′), which has a strong anisotropy at the boundary plate. Therefore, we should consider a layer in which the radiation field adjusts itself from a strongly anisotropic state at the boundary to a nearly isotropic state out of the layer. This is the radiation layer. The boundary condition at $x=0$ for Eq. (13) will be determined only by the analysis of the radiation layer.

Radiation Layer

In the radiation layer, the length scale is of the order of the photon mean free path. Therefore, we use ξ as the spatial variable. The series expansions corresponding to those in Eq. (12) are now

$$\bar{f}_i=\bar{f}_i^K(\xi)+\bar{f}_i^H(\epsilon\xi)=\bar{f}_{i0}+\epsilon\bar{f}_{i1}+\epsilon^2\bar{f}_{i2}+\epsilon^3\bar{f}_{i3}+\ldots \tag{15}$$

The exact forms of $\bar{f}_{ij}$ are shown in Eq. (A1) in the Appendix. $\bar{f}_i^K$ shows a correction term valid only in the radiation layer. This is a fast-decaying function of ξ when ξ becomes large.

Again substituting this form of series expansions into Eq. (1′)-(5′), we obtain final reduced equations. For example, the reduced equation of $\bar{T}_{i0}^K$ is

$$\bar{T}_{i0}^K-\Lambda[\bar{T}_{i0}^K]+½\bar{T}_{i0}^H(0)E_2(\xi)=(\gamma^2/p^2)F_i(\xi) \tag{16}$$

where

$$\Lambda[f(\xi)]=\frac{1}{2}\int_0^\infty f(\xi)E_1(|\xi-\xi'|)\mathrm{d}\xi' \tag{17}$$

$$F_i(\xi)=e^{-\xi} \quad \text{for } i=1$$

$$=E_2(\xi)/2 \quad \text{for } i=2 \tag{18}$$

where E_n is the exponential integral of the nth order[9] defined by

$$E_n(\xi)=\int_1^\infty \frac{e^{-\xi x'}}{x'^n}\mathrm{d}x'$$

Exact forms of all the reduced equations are shown in Eq. (A2) of the Appendix. $\bar{T}_{i0}^H(0)$ is an unknown constant with respect to ξ. This gives a boundary condition for Eq. (13).

We should point out the fact that Eq. (16) is an integral equation with respect to the spatial variable ξ. The time change is included in the equation only as a parameter. Therefore, the equation has essentially the same characters as a corresponding equation of a steady problem. The integral equations of the type of Eq. (16) were studied in previous papers[7,8] (see also Ref. 9). The equation has a character that the solution $\bar{T}_{i0}^K$ is determined simultaneously with the constant of the E_2 term, $\bar{T}_{i0}^H(0)$. In this manner, we can obtain the boundary conditions at $x=0$ for the asymptotic region as well as the solution in the radiation layer. Consequently, we can write down our solution valid to $\mathcal{O}(\epsilon^3)$ in the image space of the Laplace transformation.

Inverse Transformation

The formal inverse transformation of the image solution is

$$T_i(\xi,\tau)=\delta_{i1}[\mathcal{K}^K(\xi,\tau)+\mathcal{K}^H(\xi,\tau)]+A_i[\mathcal{G}^K(\xi,\tau)+\mathcal{G}^H(\xi,\tau)] \tag{19}$$

where

$$A_i=2A_e \quad \text{for } i=1 \qquad A_i=1 \quad \text{for } i=2$$

and δ_{i1} is the Kronecker's delta and A_e is 2.5182. (See Ref. 7.) Exact forms of $\mathcal{K}^K$, $\mathcal{K}^H$, $\mathcal{G}^K$, and $\mathcal{G}^H$ are shown in Eq. (A3) of the Appendix.

We have a function $\Theta_n(\xi,\tau)$, $n=0$-4, in the expressions of $\mathcal{K}^K$, $\mathcal{K}^H$, $\mathcal{G}^K$, and $\mathcal{G}^H$. This represents the formal inverse transform of

$$\bar{\Theta}_n(\xi)=(\gamma^2/p^2)(\epsilon\gamma)^n e^{-\epsilon\gamma\xi} \tag{20}$$

The analytic inverse transform of this function is impossible. Fortunately, when $\xi=0$, the inverse transform of Θ_n can be obtained in the analytic form as shown in Eq. (A4) in the Appendix, which in turn leads to $\mathcal{K}^K$ and $\mathcal{G}^K$. To obtain $\mathcal{K}^H$ and $\mathcal{G}^H$, the reasoning is as follows: 1) Eq. (20) shows that Θ_n is governed by a differential equation in the same form as Eq. (14); 2) $\Theta_n(0,\tau)$ provides the boundary condition; 3) $\mathcal{K}^H$ is a linear combination of $\Theta_n(\xi,\tau)$; and 4) $\mathcal{G}^H$ can be divided into two linear combinations of Θ_n, e.g., $\mathfrak{M}^H$ and $\mathfrak{N}^H$. Therefore, $\mathcal{K}^H$, $\mathfrak{M}^H$, and $\mathfrak{N}^H$ also satisfy a differential equation of the type of Eq. (14). We solve these equations of $\mathcal{K}^H$, $\mathfrak{M}^H$, or $\mathfrak{N}^H$ numerically by the finite difference method.

The finite difference scheme is an ordinary scheme of forward difference in time and centered difference in space. This scheme is very stable for wide range of values of difference meshes $\Delta\varphi$ in time and $\Delta\xi$ in space, because the scheme is an implicit one due to the last term in Eq. (14). φ is $(1-\omega)\tau/(2\omega)$. Another boundary condition is that Θ_n tend

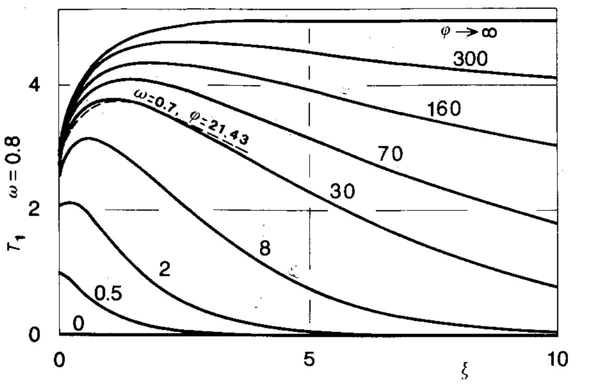

Fig. 3a Temperature profile for the beam radiation T_l for various values of φ when $\omega = 0.8$.

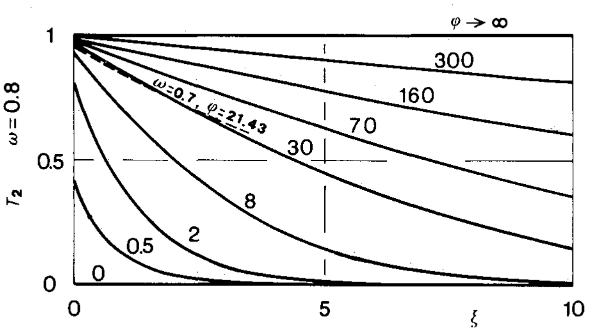

Fig. 3b Temperature profile for the blackbody radiation T_2 for various values of φ when $\omega = 0.8$.

toward zero as $\xi \to \infty$ is approximated by $\partial \Theta_n / \partial \xi = 0$ at $\xi = 20$. We choose our solution in the region where $0 \leqq \xi \leqq 10$. In most cases, $\Delta\xi$ was chosen as 0.02 and $\Delta\varphi$ as 0.01-0.5. The calculations were carried out on a FACOM M-382 computer, with a CPU time less than 1 s for every case discussed here.

Results and Discussion

In Fig. 1 the temperature changes at the boundary, $T_l(0,\tau)$ and $T_2(0,\tau)$, are shown as functions of τ for the various values of the albedo of scattering ω. From the results, the case for $0.9 \leq \omega < 1$ is found to be qualitatively different from the others. We choose a representative case of scattering-dominant medium to be that of $\omega = 0.99$. The case of $\omega = 0.8$ is chosen as a reference for a qualitatively similar case to the weak scattering one. In Fig. 2 distributions of T_l and T_2 for $\omega = 0.99$ are shown for various values of φ. The corresponding distributions for $\omega = 0.8$ are shown in Fig 3. The radiation layer is the layer where $0 \leqq \xi \lesssim 3$. When $\omega = 0.8$ in Fig. 3, T_l and T_2 are restricted in the radiation layer for low values of φ. By contrast, when $\omega = 0.99$ in Fig. 2 distributions of T_l and T_2 have long tails into the asymptotic region. When φ has a moderate or large value, the temperature distributions in Figs. 2a and 3a or in Figs. 2b and 3b have similar characteristics. In this comparison, the absolute value of φ has little meaning, since φ itself depends on ω.

The heat flux q_l and q_2 can be obtained by Eq. (1′). They are shown in Fig 4 for $\omega = 0.99$ and in Fig. 5 for $\omega = 0.8$. The heat flux through the scattering-dominant medium again has a long tail into the asymptotic region.

Finally, we compare our results to the previous findings.[6] In Fig. 3 the temperature distributions of the previous results are shown when $\omega = 0.7$ and $\varphi = 21.43$. The method of analysis in Ref. 6 is different from the present one. The case for $\omega = 0.7 \sim 0.8$ is an unfavorable case for both the present and the previous analyses. However, both results are similar qualitatively. But, quantitatively the values of φ do not coincide. This region of the value of ω may be a limit on both approximations.

We have made many assumptions to simplify the problem, many of which are not critical. For example, the external

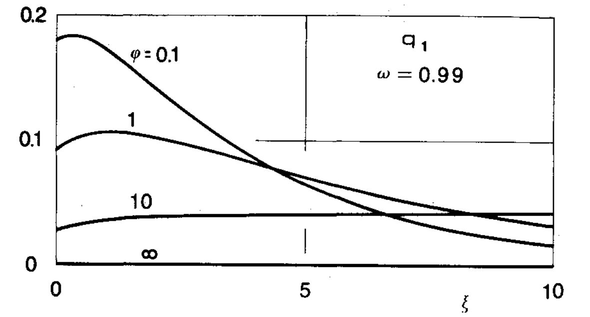

Fig. 4a Profile of heat flux for the beam radiation q_l for various values of φ when $\omega = 0.99$.

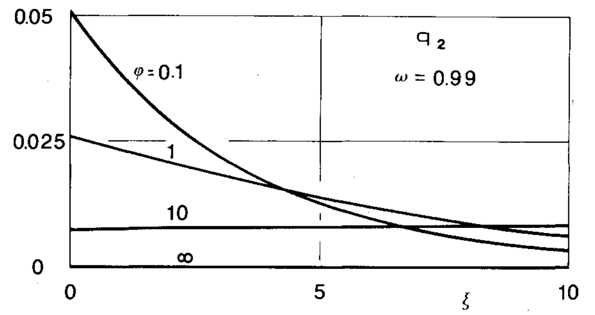

Fig. 4b Profile of heat flux for the blackbody radiation q_2 for various values of φ when $\omega = 0.99$.

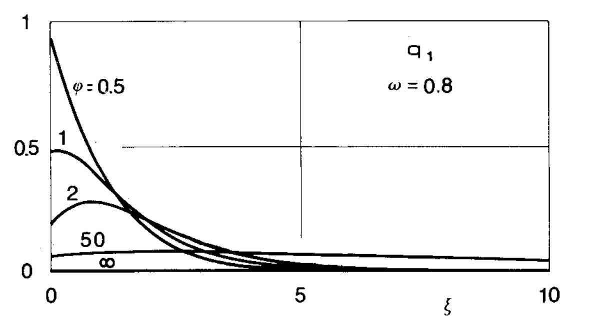

Fig. 5a Profile of heat flux for the beam radiation q_l for various values of φ when $\omega = 0.8$.

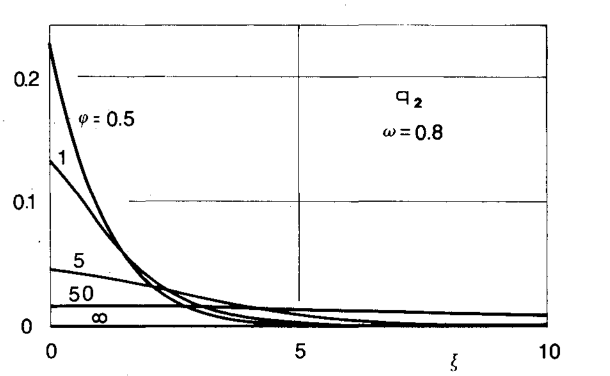

Fig. 5b Profile of heat flux for the blackbody radiation q_2 for various values of φ when $\omega = 0.8$.

beam radiation normal to the boundary plate is only one representative of a strongly anisotropic radiation input. If we remove the assumption of linearization, the essential nonlinear effect will appear only in the asymptotic region. However, the equation has the form of a purely differential equation instead of the integrodifferential equation of the original. It will not be easy to obtain an analytic solution of this nonlinear equation. But it will be easily solved numerically because it will have a character similar to the linearized equation. The restriction of the one-dimensional geometry can be removed without much difficulty in a problem whose standard length of change is much larger than the photon mean free path. It will be an almost trivial extension to remove these restrictions.

Appendix

Many complicated expressions have been used in the text. Exact forms of all of them are denoted here.

Expansions for Quantities in the Radiation Layer

We expanded the quantities in the radiation layer in Eq. (15). They have a form of

$$\bar{f}_i = \bar{f}_i^K(\xi) + \bar{f}_i^H(\epsilon\xi) = [\bar{f}_{i0}^K(\xi) + \bar{f}_{i0}^H(0)] + \epsilon\left[\bar{f}_{i1}^K(\xi) + \bar{f}_{i1}^H(0) + \xi\frac{\mathrm{d}\bar{f}_{i0}^H(0)}{\mathrm{d}x}\right] + \epsilon^2\left[\bar{f}_{i2}^K(\xi) + \bar{f}_{i2}^H(0) + \xi\frac{\mathrm{d}\bar{f}_{i1}^H(0)}{\mathrm{d}x} + \frac{1}{2}\xi^2\frac{\mathrm{d}^2\bar{f}^H(0)}{\mathrm{d}x^2}\right] + \epsilon^3\left[\bar{f}_{i3}^K(\xi) + \bar{f}_{i3}^H(0) + \xi\frac{\mathrm{d}\bar{f}_{i2}^H(0)}{\mathrm{d}x} + \frac{1}{2}\xi^2\frac{\mathrm{d}^2\bar{f}_{i1}^H(0)}{\mathrm{d}x^2} + \frac{1}{6}\xi^3\frac{\mathrm{d}^3\bar{f}_{i0}^H(0)}{\mathrm{d}x^3}\right] + \ldots \quad \text{(A1)}$$

Equations for $\bar{T}_{ij}^K$

T_{ij}^K satisfies an integral equation similar to Eq. (16),

$$\bar{T}_{i0}^K - \Lambda[\bar{T}_{i0}^K] + \frac{1}{2}\bar{T}_{i0}^H(0)E_2(\xi) = \frac{\gamma^2}{p^2}F_i(\xi) \quad \text{(A2a)}$$

$$\bar{T}_{i1}^K - \Lambda[\bar{T}_{i1}^K] + \frac{1}{2}\bar{T}_{i1}^H(0)E_2(\xi) = -\frac{\gamma}{2}\bar{T}_{i0}^H(0)E_3(\xi) \quad \text{(A2b)}$$

$$\bar{T}_{i2}^K - \Lambda[\bar{T}_{i2}^K] + \frac{1}{2}\bar{T}_{i2}^H(0)E_2(\xi) = -\frac{\gamma}{2}\bar{T}_{i1}^H(0)E_3(\xi) - \frac{\gamma^2}{2}\bar{T}_{i0}^H(0)E_4(\xi) - \frac{1}{3}\gamma^2\bar{T}_{i0}^K \quad \text{(A2c)}$$

$$\bar{T}_{i3}^K - \Lambda[\bar{T}_{i3}^K] + \frac{1}{2}\bar{T}_{i3}^H(0)E_2(\xi) = -\frac{\gamma}{2}\bar{T}_{i2}^H(0)E_3(\xi) - \frac{\gamma^2}{2}\bar{T}_{i1}^H(0)E_4(\xi) - \frac{\gamma^3}{2}\bar{T}_{i0}^H(0)E_5(\xi) - \frac{1}{3}\gamma^2\bar{T}_{i1}^K + \frac{3}{20}\gamma^3\bar{T}_{i0}^H(0)E_3(\xi) \quad \text{(A2d)}$$

Forms of $\mathcal{K}^K$, $\mathcal{K}^H$, $\mathcal{G}^K$, and $\mathcal{G}^H$

$\mathcal{K}^K$, $\mathcal{K}^H$, $\mathcal{G}^K$, and $\mathcal{G}^H$ in Eq. (19) are

$$\mathcal{K}^K(\xi,\tau) = \Theta_0(0,\tau)\mathcal{G}_e(\xi) - \frac{1}{6}\Theta_2(0,\tau)\sum_{n=3}^{N}B_{en}\mathcal{G}_n(\xi) + \frac{1}{6}\Theta_3(0,\tau)\mathcal{G}_3(\xi)\cdot\left(B_{e2} + \sum_{n=3}^{N}B_{en}S_n\right) \quad \text{(A3a)}$$

$$\mathcal{K}^H(\xi,\tau) = -\frac{2}{3}[\Theta_2(\xi,\tau) - S_3\Theta_3(\xi,\tau)]\cdot\left(B_{e2} + \sum_{n=3}^{N}B_{en}S_n\right) \quad \text{(A3b)}$$

$$\mathcal{G}^K(\xi,\tau) = -\frac{1}{2}\Theta_1(0,\tau)\mathcal{G}_3(\xi) + \frac{1}{2}\Theta_2(0,\tau)[S_3\mathcal{G}_3(\xi) - \mathcal{G}_4(\xi)] + \frac{1}{2}\Theta_3(0,\tau)\left[\left(-S_3^2 + S_4 + \frac{3}{10}\right)\mathcal{G}_3(\xi) + S_3\mathcal{G}_4(\xi) - \mathcal{G}_5(\xi) + \frac{1}{3}\sum_{n=3}^{N}B_{3n}\mathcal{G}_n(\xi)\right] \quad \text{(A3c)}$$

$$\mathcal{G}^H(\xi,\tau) = \mathfrak{M}^H(\xi,\tau) + \xi\cdot\mathfrak{N}^H(\xi,\tau) \quad \text{(A3d)}$$

where

$$\mathfrak{M}^H(\xi,\tau) = \Theta_0(\xi,\tau) - S_3\Theta_1(\xi,\tau) + (S_3^2 - S_4)\Theta_2(\xi,\tau) + \left[S_3\left(-S_3^2 + 2S_4 + \frac{3}{10}\right) - S_5 + \frac{1}{3}\left(B_{32} + \sum_{n=3}^{N}B_{3n}S_n\right)\right]\Theta_3(\xi,\tau) \quad \text{(A4)}$$

$$\mathfrak{N}^H(\xi,\tau) = \frac{3}{10}[\Theta_3(\xi,\tau) - S_3\Theta_4(\xi,\tau)] \quad \text{(A5)}$$

where $\mathcal{G}_e$ and $\mathcal{G}_n$ are universal functions independent of the parameters of the present problem and A_e, S_n, B_{en}, and B_{mn} are related constants. These functions and constants were already obtained approximately in previous papers[7,8] when $N=15$.

$\Theta_n(0,\tau)$

When $\xi=0$, the inverse transforms of Θ_n in Eq. (20) are given as

$$\Theta_0(0,\tau) = 1 - e^{-2\varphi} \quad \text{(A6a)}$$

$$\Theta_1(0,\tau) = 2\sqrt{3}\left(\frac{1-\omega}{\omega}\right)^{1/2}e^{-\varphi}[I_0(\varphi) - I_1(\varphi)] \quad \text{(A6b)}$$

$$\Theta_2(0,\tau) = 6\frac{1-\omega}{\omega}\varphi e^{-2\varphi} \quad \text{(A6c)}$$

$$\Theta_3(0,\tau) = 6\sqrt{3}\left(\frac{1-\omega}{\omega}\right)^{3/2}\varphi\left\{I_0(\varphi) - \frac{1}{3}I_1(\varphi) - \frac{4}{3}[I_0(\varphi) - I_1(\varphi)]\right\} \quad \text{(A6d)}$$

$$\Theta_4(0,\tau) = 18\left(\frac{1-\omega}{\omega}\right)^2\varphi(1-\varphi)e^{-2\varphi} \quad \text{(A6e)}$$

where I_0 and I_1 are, respectively, the modified Bessel functions of the first kind of order zero and one.

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